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Note

# Strong limit theorems for arbitrary stochastic sequences ☆

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**Abstract**

In this paper, we establish two strong limit theorems for arbitrary stochastic sequences. As corollaries, we generalize some known results.


*Keywords:* Strong law of large numbers; Stochastic sequences; Martingale difference sequences

## 1. Introduction

Let $\{X_n, \mathcal{F}_n,\ n \geqslant 0\}$ be a stochastic sequence on the probability space $(\Omega, \mathcal{F}, P)$, that is, the sequence of $\sigma$-fields $\{\mathcal{F}_n,\ n \geqslant 0\}$ in $\mathcal{F}$ is increasing in $n$ (that is $\mathcal{F}_n \uparrow$), and $X_n$ is measurable $\mathcal{F}_n$.

In Ref. [3], Jardas et al. have proved a strong law of large numbers for sequences of independent random variables which generalized Chung's classical strong law of large numbers (see [1, p. 124]). In Ref. [4], Liu and Yang have proved two strong limit theorems for arbitrary stochastic sequences which generalized Chow's strong law of large numbers for martingale difference sequences (see [2, p. 35]) and also Chung's classical strong law of large numbers.

In this paper, we establish two more general strong limit theorems for arbitrary stochastic sequences. As corollaries, we generalize Jardas et al.'s result for sequences of independent random variables as well as Liu and Yang's results for arbitrary stochastic sequences.

We first give a lemma.

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**Lemma 1.** (Doob, see [2, p. 33].) *Let* $\{X_n, \mathcal{F}_n,\ n \geqslant 0\}$ *be a martingale difference sequence. Then* $S_n = \sum_{k=1}^n X_k$ *converges a.e. on the set* $\{\sum_{k=1}^\infty E[X_k^2 \mid \mathcal{F}_{k-1}] < \infty\}$.

## 2. Main results

**Theorem 1.** *Let* $\{X_n, \mathcal{F}_n,\ n \geqslant 0\}$ *be a stochastic sequence defined as before, and* $\{c_n,\ n \geqslant 1\}$ *be a sequence of non-zero random variables such that* $c_n$ *is measurable* $\mathcal{F}_{n-1}$. *Let* $\varphi_n : R_+ \to R_+$ *be Borel functions and let* $\alpha_n \geqslant 1,\ \beta_n \leqslant 2,\ K_n \geqslant 1$ *and* $M_n \geqslant 1$ $(n \geqslant 1)$ *be constants satisfying*

$$t_1 \leqslant t_2 \quad \Rightarrow \quad \frac{\varphi_n(t_1)}{t_1^{\alpha_n}} \leqslant K_n \frac{\varphi_n(t_2)}{t_2^{\alpha_n}} \quad \text{and} \quad \frac{t_1^{\beta_n}}{\varphi_n(t_1)} \leqslant M_n \frac{t_2^{\beta_n}}{\varphi_n(t_2)}. \tag{1}$$

*Set*

$$A = \left\{\omega\colon \sum_{n=1}^\infty K_n E\big[\varphi_n\big(|X_n|\big) \mid \mathcal{F}_{n-1}\big]/\varphi_n\big(|c_n|\big) < \infty\right\}, \tag{2}$$

$$B = \left\{\omega\colon \sum_{n=1}^\infty M_n E\big[\varphi_n\big(|X_n|\big) \mid \mathcal{F}_{n-1}\big]/\varphi_n\big(|c_n|\big) < \infty\right\}. \tag{2$'$}$$

*Then*

$$\sum_{n=1}^\infty c_n^{-1}\big\{X_n - E[X_n \mid \mathcal{F}_{n-1}]\big\} \text{ converges a.e. on } AB. \tag{3}$$

**Proof.** Let $n \geqslant 0,\ X_n^* = X_n I(|X_n| \leqslant |c_n|)$. Let $k$ be a positive integral number, and let $Z_n = \varphi_n(|X_n|)/\varphi_n(|c_n|)$,

$$A_k = \left\{\omega\colon \sum_{n=1}^\infty K_n E[Z_n \mid \mathcal{F}_{n-1}] \leqslant k\right\}, \tag{4}$$

$$\tau_k = \min\left\{n\colon \sum_{i=1}^{n+1} K_i E[Z_i \mid \mathcal{F}_{i-1}] > k\right\}, \tag{5}$$

where $\tau_k = \infty$, if the right-hand side of (5) is empty. It is easy to see that $\sum_{i=1}^{\tau_k \wedge n} K_i Z_i = \sum_{i=1}^n I(\tau_k \geqslant i) K_i Z_i$. Since $I(\tau_k \geqslant i)$ is measurable $\mathcal{F}_{i-1}$, we have

$$\begin{aligned} E\left(\sum_{i=1}^{\tau_k \wedge n} K_i Z_i\right) &= E\left(\sum_{i=1}^n K_i I(\tau_k \geqslant i) Z_i\right) = E\left\{\sum_{i=1}^n K_i E\big[I(\tau_k \geqslant i) Z_i \mid \mathcal{F}_{i-1}\big]\right\} \\ &= E\left\{\sum_{i=1}^n K_i I(\tau_k \geqslant i) E[Z_i \mid \mathcal{F}_{i-1}]\right\} = E\left\{\sum_{i=1}^{\tau_k \wedge n} K_i E[Z_i \mid \mathcal{F}_{i-1}]\right\} \leqslant k. \end{aligned} \tag{6}$$

Since $A_k = \{\tau_k = \infty\}$, we have by (6), for all $n$,

$$\begin{aligned}\sum_{i=1}^{n} K_i \int_{A_k} Z_i\, dP &= \sum_{i=1}^{n} K_i E\big[I(A_k)Z_i\big] = E\Bigg\{I(A_k)\sum_{i=1}^{n} K_i Z_i\Bigg\} \\ &= E\Bigg\{I(\tau_k=\infty)\sum_{i=1}^{n} K_i Z_i\Bigg\} = E\Bigg\{I(\tau_k=\infty)\sum_{i=1}^{\tau_k\wedge n} K_i Z_i\Bigg\} \\ &\leqslant E\Bigg\{\sum_{i=1}^{\tau_k\wedge n} K_i Z_i\Bigg\} \leqslant k.\end{aligned}$$

Hence we have

$$\sum_{n=1}^{\infty} K_n \int_{A_k} Z_n\, dP \leqslant k. \tag{7}$$

By (1), we have $|x|/|c_n(\omega)| \leqslant |x|^{\alpha_n}/|c_n(\omega)|^{\alpha_n} \leqslant K_n\varphi_n(|x|)/\varphi_n(|c_n(\omega)|)$, as $|x| > |c_n(\omega)|$. Hence

$$\begin{aligned}&\sum_{n=1}^{\infty} P\big\{A_k\big(X_n^* \neq X_n\big)\big\} \\ &\quad = \sum_{n=1}^{\infty} \int_{A_k(|X_n|>|c_n|)} dP \leqslant \sum_{n=1}^{\infty} \int_{A_k(|X_n|>|c_n|)} \frac{|X_n|}{|c_n|}\, dP \\ &\quad \leqslant \sum_{n=1}^{\infty} \int_{A_k(|X_n|>|c_n|)} K_n \frac{\varphi_n(|X_n|)}{\varphi_n(|c_n|)}\, dP \leqslant \sum_{n=1}^{\infty} \int_{A_k} K_n Z_n\, dP \leqslant k.\end{aligned} \tag{8}$$

By (8) and Borel–Cantelli lemma, we have $P\{A_k(X_n^* \neq X_n),\ \text{i.o.}\} = 0$. Hence we have

$$\sum_{n=1}^{\infty} \big(X_n - X_n^*\big)/c_n \quad \text{converges a.e. on } A_k. \tag{9}$$

Since $A = \bigcup_k A_k$, we have

$$\sum_{n=1}^{\infty} \big(X_n - X_n^*\big)/c_n \quad \text{converges a.e. on } A. \tag{10}$$

Since

$$\begin{aligned}&\big|\big(E[X_n \mid \mathcal{F}_{n-1}] - E\big[X_n^* \mid \mathcal{F}_{n-1}\big]\big)/c_n\big| \\ &\quad = \big|E\big[\big(X_n - X_n^*\big)/c_n \mid \mathcal{F}_{n-1}\big]\big| \leqslant E\big[\big|X_n - X_n^*\big|/|c_n| \mid \mathcal{F}_{n-1}\big] \\ &\quad = E\big[\big(|X_n|/|c_n|\big)I\big(|X_n| > |c_n|\big) \mid \mathcal{F}_{n-1}\big] \\ &\quad \leqslant K_n E\big[\varphi_n\big(|X_n|\big)/\varphi_n\big(|c_n|\big)I\big(|X_n| > |c_n|\big) \mid \mathcal{F}_{n-1}\big] \\ &\quad \leqslant K_n E\big[\varphi_n\big(|X_n|\big) \mid \mathcal{F}_{n-1}\big]/\varphi_n\big(|c_n|\big) \quad \text{a.e.}\end{aligned} \tag{11}$$

By (11) and (2), we have

$$\sum_{n=1}^{\infty} \big(E[X_n \mid \mathcal{F}_{n-1}] - E\big[X_n^* \mid \mathcal{F}_{n-1}\big]\big)/c_n \quad \text{converges a.e. on } A. \tag{12}$$

Let $Y_0 \equiv 0$ and

$$Y_n = \{X_n^* - E[X_n^* \mid \mathcal{F}_{n-1}]\}/c_n, \quad n \geqslant 1. \tag{13}$$

It is clear that $\{Y_n, \mathcal{F}_n, n \geqslant 0\}$ is a martingale difference sequences. Observe that

$$\begin{aligned} E[Y_n^2 \mid \mathcal{F}_{n-1}] &= \{E[(X_n^*)^2 \mid \mathcal{F}_{n-1}] - (E[X_n^* \mid \mathcal{F}_{n-1}])^2\}/c_n^2 \\ &\leqslant E\left[\left(\frac{X_n^*}{c_n}\right)^2 \,\middle|\, \mathcal{F}_{n-1}\right] \quad \text{a.e.} \end{aligned} \tag{14}$$

By (1), we have $x^2/c_n^2(\omega) \leqslant |x|^{\beta_n}/|c_n(\omega)|^{\beta_n} \leqslant M_n \varphi_n(|x|)/\varphi_n(|c_n(\omega)|)$, if $|x| \leqslant |c_n(\omega)|$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} E[Y_n^2 \mid \mathcal{F}_{n-1}] &\leqslant \sum_{n=1}^{\infty} E\left[\left(\frac{X_n^*}{c_n}\right)^2 \,\middle|\, \mathcal{F}_{n-1}\right] \\ &= \sum_{n=1}^{\infty} E\left[\left(\frac{X_n}{c_n}\right)^2 I(|X_n| \leqslant |c_n|) \,\middle|\, \mathcal{F}_{n-1}\right] \\ &\leqslant \sum_{n=1}^{\infty} M_n E\left[\frac{\varphi(|X_n|)}{\varphi(|c_n|)} I(|X_n| \leqslant |c_n|) \,\middle|\, \mathcal{F}_{n-1}\right] \\ &\leqslant \sum_{n=1}^{\infty} M_n E\left[\frac{\varphi(|X_n|)}{\varphi(|c_n|)} \,\middle|\, \mathcal{F}_{n-1}\right] < \infty \quad \text{a.e.,}\ \omega \in B. \end{aligned} \tag{15}$$

By Lemma 1,

$$\sum_{n=1}^{\infty} Y_n = \sum_{n=1}^{\infty} \{X_n^* - E[X_n^* \mid \mathcal{F}_{n-1}]\}/c_n \quad \text{converges a.e. on } B. \tag{16}$$

(3) follows from (10), (12) and (16). The proof of this theorem is completed. □

**Corollary 1.** (See [4].) *Let $\{X_n, \mathcal{F}_n, n \geqslant 0\}$ be defined as in Theorem 1, $\{a_n, n \geqslant 1\}$ be a sequence of positive random variables such that $a_n$ is measurable $\mathcal{F}_{n-1}$. Let $\{f_n(x), n \geqslant 1\}$ be a sequence of non-negative even functions on $R$ and positive in the interval $x > 0$ such that $f_n(x)/x$ and $x^2/f_n(x)$ are non-decreasing in the interval $x > 0$. Set*

$$A = \left\{\omega \colon \sum_{n=1}^{\infty} E[f_n(X_n) \mid \mathcal{F}_{n-1}]/f_n(a_n) < \infty\right\}. \tag{17}$$

*Then*

$$\sum_{n=1}^{\infty} a_n^{-1}\{X_n - E[X_n \mid \mathcal{F}_{n-1}]\} \quad \textit{converges a.e. on } A. \tag{18}$$

**Proof.** Letting $\varphi_n(x) = f_n(|x|) = f_n(x)$, $c_n = a_n$, $\alpha_n = 1$, $\beta_n = 2$, $K_n = 1$ and $M_n = 1$ for all $n$ in Theorem 1, this corollary follows. □

This corollary extends Chow's strong law of large numbers for martingale difference sequences as follows:

**Corollary 2.** (See [2, p. 35].) *Let $\{X_n, \mathcal{F}_n,\ n \geqslant 0\}$ be a martingale difference sequence, and let $\{a_n\}$ be defined as in Corollary 1. If $1 \leqslant p \leqslant 2$, then $\sum_{n=1}^{\infty} a_n^{-1} X_n$ converges a.e. on the set $A = \{\sum_{n=1}^{\infty} a_n^{-p} E[|X_n|^p \mid \mathcal{F}_{n-1}] < \infty\}$.*

**Corollary 3.** *Let $\{X_n, \mathcal{F}_n,\ n \geqslant 0\}$, $\{\varphi_n(x),\ n \geqslant 1\}$, $K_n$ and $M_n$ be given as in Theorem 1, and $\{c_n,\ n \geqslant 1\}$ be a sequence of non-zero numbers. If*

$$\sum_{n=1}^{\infty} K_n \frac{E\varphi_n(|X_n|)}{\varphi_n(|c_n|)} < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} M_n \frac{E\varphi_n(|X_n|)}{\varphi_n(|c_n|)} < \infty, \tag{19}$$

*then*

$$\sum_{n=1}^{\infty} c_n^{-1}\big\{X_n - E[X_n \mid \mathcal{F}_{n-1}]\big\} \quad \textit{converges a.e.} \tag{20}$$

**Proof.** By (19) and non-negativity of $\varphi_n$, $K_n$ and $M_n$ for all $n$, we have

$$\sum_{n=1}^{\infty} K_n \frac{E[\varphi_n(|X_n|) \mid \mathcal{F}_{n-1}]}{\varphi_n(|c_n|)} < \infty \quad \text{a.e.},$$
$$\sum_{n=1}^{\infty} M_n \frac{E[\varphi_n(|X_n|) \mid \mathcal{F}_{n-1}]}{\varphi_n(|c_n|)} < \infty \quad \text{a.e.}$$

Thus $P(A) = P(B) = 1$. This corollary follows from Theorem 1. □

**Corollary 4.** (See [3].) *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of independent random variables with $EX_n = 0$ for all $n$, and let $\{c_n,\ n \geqslant 1\}$ be a sequence of non-zero numbers. Let $\{\varphi_n(x),\ n \geqslant 1\}$, $K_n$ and $M_n$ be given as in Theorem 1. If (19) holds, then*

$$\sum_{n=1}^{\infty} c_n^{-1} X_n \quad \textit{converges a.e.} \tag{21}$$

From Corollary 1 or Corollary 4, we can easily obtain Chung's strong law of large numbers for sequences of independent random variables (see [1, p. 124]) as follows:

**Corollary 5.** *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of independent random variables with $EX_n = 0$ for all $n$, and let $\{c_n,\ n \geqslant 1\}$ be a positive increasing sequence of real numbers tending to infinity. Let $f : R \to R$ be a positive, even function such that $f(x)/x$ and $x_2/f(x)$ are non-decreasing for $x > 0$. If $\sum_{n=1}^{\infty} Ef(X_n)/f(c_n) < \infty$, then $\sum_{n=1}^{\infty} c_n^{-1} X_n$ converges a.e.*

**Theorem 2.** *Let $\{X_n, \mathcal{F}_n,\ n \geqslant 0\}$ and $\{c_n,\ n \geqslant 1\}$ be given as in Theorem 1. Let $\varphi_n(x) : R_+ \to R_+$ be Borel functions and let $\alpha_n \geqslant 0$, $\beta_n \leqslant 1$, $K_n \geqslant 1$ and $M_n \geqslant 1$ $(n \geqslant 1)$ be constants satisfying*

$$t_1 \leqslant t_1 \quad \Rightarrow \quad \frac{\varphi_n(t_1)}{t_1^{\alpha_n}} \leqslant K_n \frac{\varphi_n(t_2)}{t_2^{\alpha_n}} \quad \textit{and}$$
$$\frac{t_1^{\beta_n}}{\varphi_n(t_1)} \leqslant M_n \frac{t_2^{\beta_n}}{\varphi_n(t_2)}. \tag{22}$$

*Let $A$, $B$ be defined as* (2) *and* (2′)*, respectively. Then*

$$\sum_{n=1}^{\infty} c_n^{-1} X_n \quad \textit{converges a.e. on } AB. \tag{23}$$

**Proof.** Let $n \geqslant 0$, $X_n^* = X_n I(|X_n| \leqslant |c_n|)$. Let $A_k$, $\tau_k$ and $Z_n$ be defined as in the proof of Theorem 1. Using a similar argument, we also prove (7) holds. By (22), we have $|x|^{\alpha_n}/|c_n(\omega)|^{\alpha_n} \leqslant K_n \varphi_n(|x|)/\varphi_n(|c_n(\omega)|)$, as $|x| > |c_n(\omega)|$. By (7), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\bigl\{A_k\bigl(X_n^* \neq X_n\bigr)\bigr\} \\
&\quad = \sum_{n=1}^{\infty} \int_{A_k(|X_n|>|c_n|)} dP \leqslant \sum_{n=1}^{\infty} \int_{A_k(|X_n|>|c_n|)} \frac{|X_n|^{\alpha_n}}{|c_n|^{\alpha_n}}\, dP \\
&\quad \leqslant \sum_{n=1}^{\infty} \int_{A_k(|X_n|>|c_n|)} K_n \frac{\varphi_n(|X_n|)}{\varphi_n(|c_n|)}\, dP \leqslant \sum_{n=1}^{\infty} \int_{A_k} K_n Z_n\, dP \leqslant k.
\end{aligned} \tag{24}$$

By (24), we similarly have (9) holds. Hence (10) holds. Let

$$B_k = \left\{\omega\colon \sum_{n=1}^{\infty} M_n E[Z_n \mid \mathcal{F}_{n-1}] \leqslant k\right\}, \tag{25}$$

$$\tau_k' = \min\left\{n\colon \sum_{i=1}^{n+1} M_i E[Z_i \mid \mathcal{F}_{i-1}] > k\right\}. \tag{26}$$

Using a similar argument used to derive (7), we have

$$\sum_{n=1}^{\infty} M_n \int_{B_k} Z_n\, dP \leqslant k. \tag{27}$$

By (22), we have $|x|/|c_n(\omega)| \leqslant |x|^{\beta_n}/|c_n(\omega)|^{\beta_n} \leqslant M_n \varphi_n(|x|)/\varphi_n(|c_n(\omega)|)$, as $|x| \leqslant |c_n(\omega)|$. Then

$$\begin{aligned}
\int_{B_k} \left(\sum_{n=1}^{\infty} \frac{|X_n^*|}{|c_n|}\right) dP &= \sum_{n=1}^{\infty} \int_{B_k} \frac{|X_n^*|}{|c_n|}\, dP \\
&= \sum_{n=1}^{\infty} \int_{B_k(|X_n| \leqslant |c_n(\omega)|)} \frac{|X_n|}{|c_n|}\, dP \\
&\leqslant \sum_{n=1}^{\infty} \int_{B_k(|X_n| \leqslant |c_n(\omega)|)} M_n \frac{\varphi_n(|X_n|)}{\varphi_n(|c_n|)}\, dP \\
&\leqslant \sum_{n=1}^{\infty} \int_{B_k} M_n \frac{\varphi_n(|X_n|)}{\varphi_n(|c_n|)}\, dP \leqslant k.
\end{aligned} \tag{28}$$

Hence $\sum_{n=1}^{\infty} \frac{X_n^*}{c_n}$ converges a.e. on $B_k$. Since $B = \bigcup_k B_k$, then

$$\sum_{n=1}^{\infty} \frac{X_n^*}{c_n} \quad \text{converges a.e. on } B. \tag{29}$$

From (10) and (28), (23) follows. $\Box$

**Corollary 6.** (See [4].) *Let* $\{X_n,\ \mathcal{F}_n,\ n \geqslant 0\}$ *and* $\{a_n,\ n \geqslant 1\}$ *be given as in Corollary* 1. *Let* $\{f_n(x), n \geqslant 1\}$ *be a sequence of non-negative functions on* $R$ *and positive in the interval* $x > 0$ *such that* $f_n(x)$ *and* $x/f_n(x)$ *are non-decreasing for* $x > 0$. *Let* $A$ *be defined by* (17). *Then*

$$\sum_{n=1}^{\infty} \frac{X_n}{c_n} \quad \textit{converges a.e. on } A. \tag{30}$$

**Proof.** Letting $\varphi_n(x) = f_n(x)$ for $x > 0$, $\alpha_n = 0$, $\beta_n = 1$, $K_n = 1$ and $M_n = 1$ in Theorem 2, this corollary follows. $\Box$

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